Dynamical origin of spatial order

M. R. Sarkardei
Physics Department, Al-Zahra University, Teheran 19934, Iran
and Mathematics Department, Imperial College, London SW7 2BZ, United Kingdom

R. L. Jacobs
Mathematics Department, Imperial College, London SW7 2BZ, United Kingdom
(Received 22 September 1994)

We present a numerical study of a one-dimensional version of the Burridge-Knopoff model [Bull. Seismol. Soc. Am. **57**, 341 (1967)] with stick-slip dynamics. The solutions of the model in the low velocity regime represent earthquakes in a simple transform fault and have chaotic behavior [J. M. Carlson and J. S. Langer, Phys. Rev. Lett. **62**, 2632 (1989); Phys. Rev. A **40**, 6470 (1989)]. It has been shown recently that in a higher velocity regime there are solutions of the model with periodic boundary conditions that are solitonlike and not necessarily chaotic [J. Schmittbuhl, J. P. Vilotte, and S. Roux, Europhys. Lett. **21**, 374 (1993)]. We show here that stable, nearly periodic solutions also exist in a certain window of parameter space when the model has free boundary conditions. These solutions are periodic in both time and space and display striation effects that are strikingly similar to those seen experimentally by Gollub and co-workers. [Phys. Rev. A **43**, 811 (1991); Phys. Rev. E **47**, 820 (1993)]. For an arbitrary disordered set of initial conditions, the short-time behavior is noisy, but the stable nearly periodic solutions emerge in the long-time limit. We discuss the origin of the window and show that the nature of the solution found depends strongly on the boundary condition. We also discuss the effects of symmetry breaking and disorder and show that even in a highly disordered regime the system can spontaneously organize itself so that very nearly stable noise-free solutions emerge.

PACS number(s): 05.40.+j, 91.30.−f, 64.60.Cn, 62.30.+d

I. INTRODUCTION

Burridge and Knopoff [1] introduced a dynamical model of plate tectonics in which the material between two plates is represented by an elastically coupled array of slider blocks. The blocks are coupled elastically to one plate and coupled to the other plate via a stick-slip friction function. The elastically coupled plate moves with constant velocity v and the other plate is fixed. A more recent one-dimensional version of the model has been discussed by Carlson and Langer [2] in which the friction function for a given slider depends on the velocity of the slider relative to the fixed plate and decreases with this velocity. They demonstrated that this homogeneous model had chaotic solutions and appeared to behave in the fashion usually described by the phrase "self-organized criticality." In their numerical solution slipping events take place in an apparently random fashion and the number of particles involved in each event varied over the whole range from 1 to N, where N is the total number of particles in the system. They used *free boundary conditions* and their model is embodied in the equations

$$m\frac{d^2X_j}{dt^2} = k_c(X_{j+1} - 2X_j + X_{j-1}) - k_p(X_j - vt) + F(\dot{X}_j), \quad (1)$$

$$m\frac{d^2X_1}{dt^2} = k_c(X_2 - X_1) - k_p(X_1 - vt) + F(\dot{X}_1), \quad (2)$$

$$m\frac{d^2X_N}{dt^2} = k_c(-X_N + X_{N-1}) - k_p(X_N - vt) + F(\dot{X}_N). \quad (3)$$

The nonlinear friction function $F(\dot{X}_j)$ for slider j is defined as follows. It balances the sum of the elastic forces on slider j if (i) the modulus of the sum is less than a threshold F_0 *and* (ii) $\dot{X}_j$, the velocity of slider j, is zero. If the sum lies outside this range *and* $\dot{X}_j$ is zero, then the sum is reduced by $\pm F_0$. Finally, if the velocity $V = \dot{X}_j$ of the slider is nonzero, then the friction function is given by

$$F(V) = -\frac{F_0}{1+ \mid V \mid}\mathrm{sgn}(V). \quad (4)$$

Schmittbuhl *et al.* [3] have discussed the same model with *periodic boundary conditions* (i.e., on a circular chain) over a different range of values of the parameters and found that a soliton mode is possible and they have also discussed the energetics which govern the appearance of this mode. They found that if the power input from the moving plate is sufficient to balance the power dissipated by a soliton mode, then the mode appears

1063-651X/95/51(3)/1929(6)/$06.00

after the system matures. A similar statement can be made about multiple soliton modes. Any power input in excess of that dissipated by solitons is dissipated chaotically. The soliton mode has a characteristic spatial width and a characteristic velocity. There is a question about whether their parameters and boundary conditions realistically describe a tectonic system. Nevertheless, their result is of considerable interest and may have application to other physically important observable phenomena.

In this paper we discuss the same model in a different parameter range with free boundary conditions which are probably more realistic. We find that there is a narrow window in parameter space in which the system settles down to a form of behavior that is periodic in time (or nearly so) and spatially ordered, independent of the conditions defining the initial positions and velocities of the sliders. This spatial ordering is carried by waves which propagate in from the ends of the chain; it is therefore not surprising that Schmittbuhl *et al.* did not observe such a mode for their system has no ends. The nature of the boundary conditions on the ends of the chain has a strong effect on the nature of the solution. This is quite an interesting point because it is usually assumed that the effects of the boundary conditions decay into the bulk. While the solution is periodic (or nearly so) in both space and time, it is not a normal mode of the system; indeed it can be shown that at any instant most of the sliders have zero velocity. The spatial ordering is strongly reminiscent of the striation effects seen in some experimental systems by Rubio *et al.* [4] and Vallette and Gollub [5] and it may have some relevance to striation phenomena which appear in some other dynamically driven systems. Examples of these are cloud patterns seen on the lea side of mountain ranges, variations in luminescence in badly adjusted fluorescent tubes, and the flow of congested traffic behind a traffic light. The system also has an interesting adaptive behavior and seems under many circumstances to select a mode of motion which is adapted to minimize energy expenditure for the given set of parameters.

II. RESULTS AND DISCUSSION

The method of solution is numerical. We use a version of the simplest Runge-Kutta scheme which is adapted to deal with the discontinuity in the friction function. The method cannot be expected to give accurate solutions over a long time span because, for example, slowly accumulating errors in the phase when the solution is periodic can give large phase shifts. Also small errors can eventually have large effects when the solution is in the chaotic regime. But over short time spans (which are nevertheless long compared with the periodicity) and for statistical properties the method should be reliable. This has been verified by various internal checks such as changing the time step and varying the initial conditions. These leave most of the relevant properties unchanged. (Those that do change will be pointed out below.) We have also checked our method by comparing our results with those previously found by Carlson and Langer [2] and by Schmittbuhl *et al.* [3]. The results are the same in all relevant respects.

One important quantity is the total elastic force acting on the chain $P(t)$. This is the sum of all the terms proportional to k_p in Eqs. (1)–(3) above. The terms proportional to k_c add up to zero. The force trace $P(t)$ is presented in Fig. 1 for the parameter set shown in the caption with different values of F_0. The same set of initial conditions is used in each case, but these conditions are extreme and disordered and the system is initially far from equilibrium. It can be seen that if $F_0 = 15$ or 40 the force trace is always noisy, but the width of the trace eventually settles down to a relatively small value when the influence of the extremely unrepresentative initial conditions decays. (For more extreme values of F_0 the width remains large.) For the remaining values of F_0 the initial noisy behavior is of limited duration and the force trace eventually settles down to a nearly periodic form. The real part of the Fourier transform of the force trace $\Pi(\omega)$ illustrates the point nicely in Figs. 2 and 3 for $F_0 = 15$ and 30, which are reasonably representative values. If $F_0 = 15$, then we have an extremely noisy Fourier transform with no sharp features. If $F_0 = 30$, then we have a Fourier transform with a sharp feature near $\omega = 6$ units and other sharp features at the harmonics. There is also a small amount of noise centred near $\omega = 3$ units. The nearly periodic behavior persists for as long as we are able to run the program, i.e., for a very long time indeed. Some quantities are independent of the initial conditions and the time step of the integration. These are the sharp frequency and the mean value of the force trace in the steady region, but the amount of noise and the width of the force trace and the phase depend somewhat on these variables. The window in which the periodic solutions appear is well defined with sharp edges at values of F_0 of 16.6 and 37. We also find that the range of values of the driving velocity v in which the phenomenon can be observed lies between $v = 0.42$ and 0.72 when $F_0 = 20$ and all other parameters remain fixed.

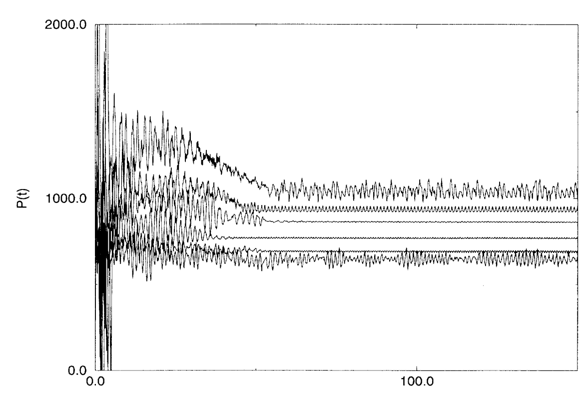

FIG. 1. Some force traces $P(t)$ plotted against time t for the parameters $N = 100$, v=0.6, k_c=40, k_p=50, and F_0=15, 17, 20, 25, 30, and 40, starting with F_0=15 on the lowest curve and ending with F_0=40 on the highest.

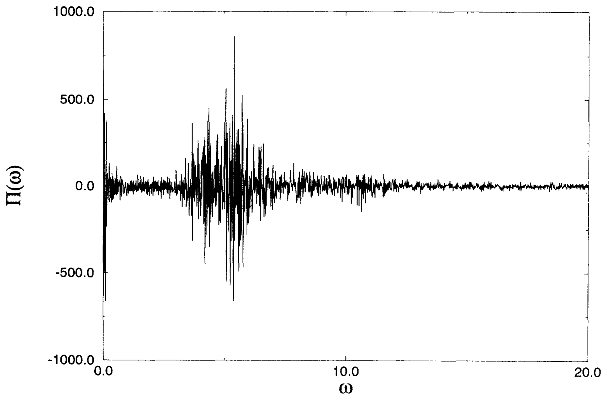

FIG. 2. Fourier transform $\Pi(\omega)$ of the force trace for $F_0=15$. Note the lack of structure and the large amplitude of the noise.

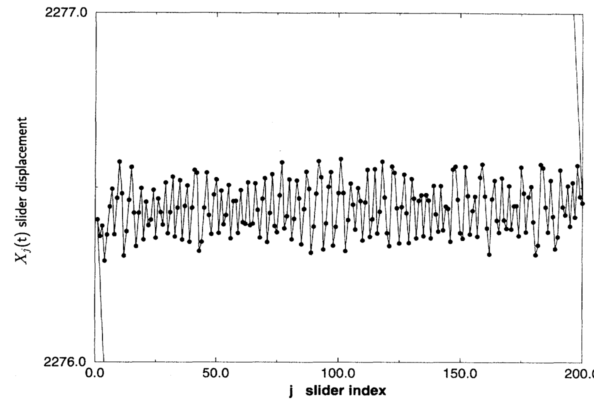

FIG. 4. Configuration of the chain after a long time when $F_0=15$ and $N=200$. The line is a guide to the eye and longitudinal displacements are plotted laterally for clarity of presentation.

We try to understand the reason for these phenomena by examining snapshots of the positions of the sliders. In Figs. 4 and 5 the configuration of the chain is shown for $F_0 = 15$ and 20 at a late time when the trace has settled down. Longitudinal displacements are plotted laterally for clarity of presentation. No spatial order is discernible for $F_0 = 15$, but for $F_0 = 20$ the chain is in a highly ordered state with a disjunction at position 86. There *appear* to be *four* waves passing down the chain from each end meeting and annihilating at position 86, i.e., there appear to be *eight* waves altogether. This is a moiré effect due to the fact that the displacement is presented at the discrete atomic positions only. There are actually only two carrier waves: one, a wave of compression moving inwards from the left, and the other, a wave of dilation moving inwards from the right. We can distinguish between waves of compression and dilation in this situation by contrast with elastic waves: in the rightward moving compressional wave the sliders on the downward slope to the right of a crest are moving to the right whereas the remaining sliders are still; however, in the leftward moving dilational wave the sliders on the upward slope to the left of a crest are moving to the right also whereas the remaining sliders are still. In other words, both waves carry matter to the right in the same direction as the driving velocity. Most of these features are independent of the initial conditions and the details of the method of integration. Only the position at which the two inward moving waves collide and their relative phases vary as we change these factors. The relative phase of the two waves at the point of collision governs the amount of noise and the width of the force trace in the long time region. If the waves meet nicely in phase then there is very little noise and the width is extremely narrow. These results are absolutely characteristic of the solutions in the parameter region examined and we have found that they hold in all of the many tens of cases examined. The situation is also illustrated in Fig. 6 where the positions are plotted at two successive instants of time.

A clue to understanding these results can be found in

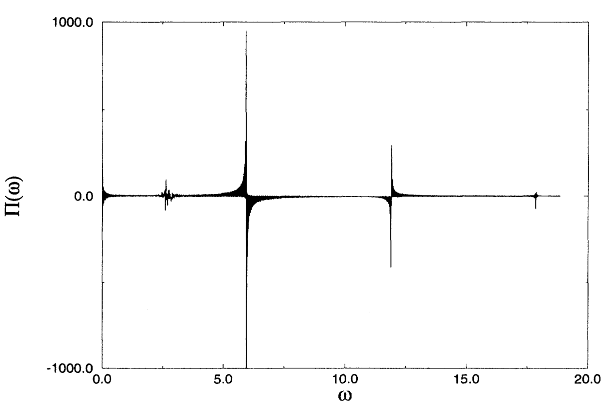

FIG. 3. Fourier transform $\Pi(\omega)$ of the force trace for $F_0=30$. Note the clear structure and the small amplitude of the noise.

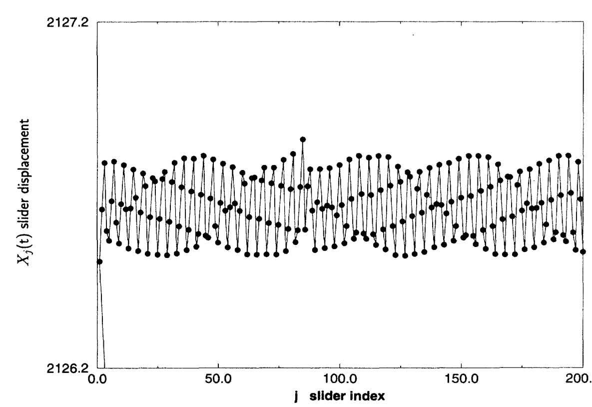

FIG. 5. Configuration of the chain after a long time when $F_0=20$ and $N=200$. The point of disjunction is at site 86 and always remains there.

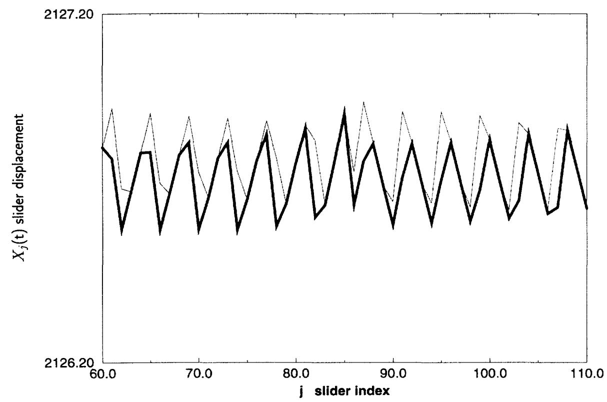

FIG. 6. Central portion of the chain at two successive late time instants. The convergence of the compressional and dilational waves on the point of disjunction can clearly be seen.

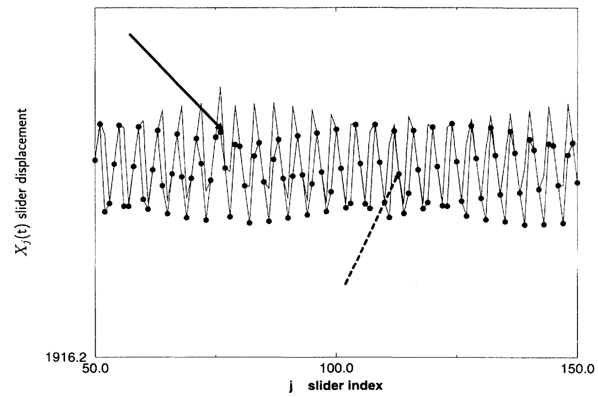

FIG. 8. Central portion of the chain at two successive late time instants. An impurity with F_0=30 is placed at site 113 (marked with a dashed arrow) and the value of F_0 elsewhere is 20. The point of disjunction is marked with a solid arrow. The change of phase at the impurity site can be seen clearly.

some snapshots of the positions of the sliders starting shortly after the beginning of a run. These are shown in Fig. 7 and it can be seen that the waves start at the ends of the chain and move to the center, thus taming the chaotic region in the middle of the chain. The instant at which they collide coincides with the instant at which the force trace becomes quiet. The situation is that the wave is stabilized by the interaction of the sliders in the bulk of the chain and the end sliders which have a different environment. If the frequency of the oscillations of the end sliders matches a frequency with which a wave can propagate down the chain, then the phenomenon can be observed. This condition defines the window in parameter space in which the phenomenon occurs. This statement can be verified by forcing the end sliders to move with whatever frequency we wish and by observing the subsequent motion of the chain. The window in parameter space in which the phenomenon occurs is now wider, corresponding to the fact that we no longer have to rely on the free oscillations of the end sliders to generate the waves. This actually enables us to find the window in which regular waves can propagate down the chain. The importance of the end conditions is now clear and it is now obvious why the phenomenon cannot be seen in a chain with periodic boundary conditions.

We now investigate the effect of broken symmetry on the results. We first change the value of F_0 on one site from the otherwise uniform value of 20. If F_0 on the

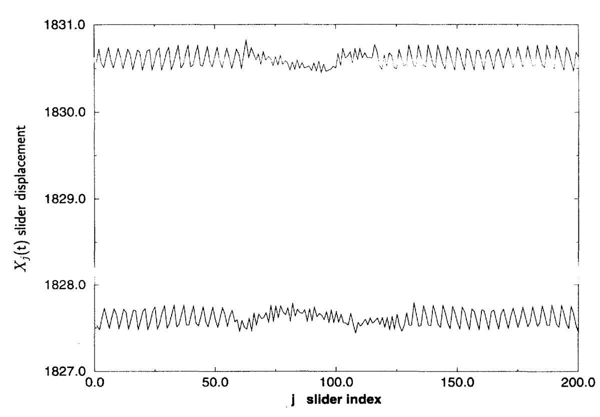

FIG. 7. Configuration of the chain at a two successive early times when F_0=20 and N=200. This shows the convergence of the two waves on the central chaotic region and its destruction.

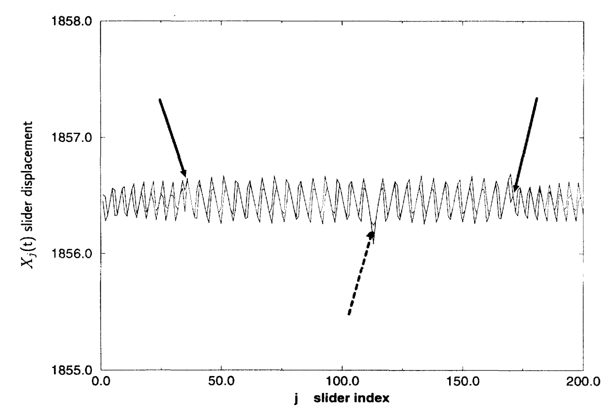

FIG. 9. The chain at two successive intermediate times. An impurity with F_0=40 is placed at site 113 (marked with a dashed arrow) and the value of F_0 elsewhere is 20. The impurity site acts as the source of two outward-traveling waves and these meet the waves coming in from the ends of the chain at the points marked by solid arrows. At these times the force trace displays beats. At later times the outward-traveling waves overcome the inward-traveling waves and reach the ends of the chain. At these later times the force trace is quiet.

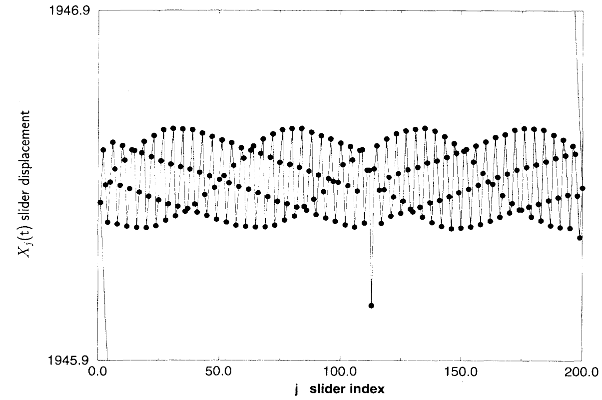

FIG. 10. The chain at a late time. An impurity with $F_0=50$ is placed at site 113 and the value of F_0 elsewhere is 20. Two inward-traveling waves meet and annihilate at the impurity site.

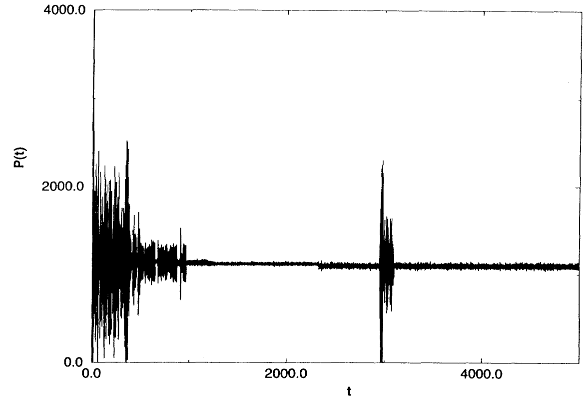

FIG. 12. Force trace for a chain with a disordered array of theshold parameters $F_0(j)$. The trace is quiet for *long* time intervals interrupted by short noisy intervals.

site in question is less than 35, then our results are unchanged, except that one inward-traveling wave passes through the disturbed site with a change of phase and the two inward-traveling waves collide at some other site, which is determined by the initial conditions (see Fig. 8). If, on the other hand, F_0 lies between 35 and 45, the oscillations of the disturbed site act as a source of outward-traveling waves which propagate to the ends of the chain, thus overcoming the inward-traveling waves which originate at the ends (see Fig. 9). In both of these cases the force trace becomes quiet. However, if $F_0 > 45$, then we have two inward-traveling waves, neither of which passes through the disturbed site. If the right-traveling wave reaches the disturbed site first, then it ends there and there is a disordered region to the right of the site which shrinks when the left-traveling wave reaches the site. Thus the two waves ultimately collide at this site and the force trace becomes quieter (see Fig. 10). These phenomena are observed for all possible positions of the disturbed site so that in a long chain a simple broken symmetry is not sufficient to ensure completely chaotic behavior and a quiet force trace can be obtained in situations of broken symmetry. There is clearly quite a rich variety of phenomena which may be reflected in the behavior of some experimental systems.

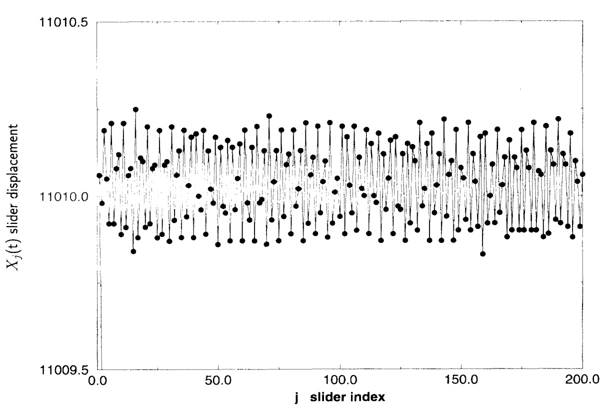

FIG. 11. Configuration of the chain with a disordered array of threshold parameters $F_0(j)$ at a time instant when the force trace is quiet. The plate velocity v is 2.75 and the chain stiffness k_c is 100. The configuration is significantly more ordered than it is at a time when the force trace is noisy.

A more serious degree of disorder has more drastic effects. We have varied $F_0(j)$ randomly from site to site so that the mean is 27 and the standard deviation is 8.5. We find that under the normal operating conditions where v is 0.6 and all the other parameters have the same values as before, the resulting force trace $P(t)$ is extremely broad and noisy at all times. If we increase the value of v to 2.75 and increase k_c to 100, then we find another window where the force trace settles down to rather stable, quiet behavior after a long time, but it has no significant periodic component. The quiet behavior seems to be associated with a hand-over-fist ladder-climbing mode of motion of *all* the sliders in the chain, but is not yet further understood (see Fig. 11). If only a small number of the sliders behave in a different fashion then the noise is much larger. These results seem to be independent of the initial conditions and the exact choice of a random configuration of $F_0(j)$ on the chain. The time taken for the force trace to settle down is very variable however and the distribution of these times considered as random variables might be interesting and provide a clue to some long-time nonexponential decay phenomena. The quiet solution is stable for very long times, but can eventually become unstable again (see Fig. 12). The reason for the disruption in the stability is not understood. If we increase the value of v further then the force trace becomes noisy again at all times. It seems to us remarkable that in chains with such a large degree of disorder the system can organize its motion *spontaneously* so as to move quietly. It would obviously be interesting to see if there are any experimental systems with this behavior. There may also be wider implications of this type of behavior.

III. CONCLUSION

We have carried out a numerical study of a one-dimensional Burridge-Knopoff N-site chain with stick-slip dynamics and free end conditions. We have shown that there are circumstances in which stable, ordered, oscillatory solutions which are relatively noiseless emerge spontaneously when chaotic solutions are expected. These stable solutions originate in the oscillations of the ends of the chain which drive waves of compression or dilation into the center. When the waves meet they annihilate and result in the stable solution. The window in parameter space in which this can occur is determined by a matching condition in which the frequency of the oscillations of the ends matches the frequency with which an oscillatory wave can propagate down the chain.

We have also investigated the effect of breaking the symmetry by disturbing the value of one parameter at one unsymmetrically placed site. We find that the stable, ordered, oscillatory solutions again emerge spontaneously, but the configuration of the chain responds in a fashion that can be relatively easily understood. The emergence of these stable, quiet solutions is possibly surprising in view of the contrasting behavior of a two-site chain which behaves chaotically when the symmetry is broken by disturbing one site [6].

We have finally investigated the behavior of a disordered chain with a random array of threshold parameters $F_0(j)$. It is very remarkable that in this case also quiet, rather stable, nearly ordered solutions emerge in certain parameter ranges.

We also note the similarity between our results and the experimental results of Gollub and co-workers [4,5] where local disturbances propagate in successive pulses down the system. Their results are not periodic or noiseless in the parameter region examined experimentally, but there is some indication of a characteristic frequency in Fig. 9 of Ref. [4]. It might be worth examining a wider region and it might also be worth disturbing the ends of their system periodically in the fashion described in the present text to see whether results comparable to ours can be obtained.

Another interesting feature is the similarity of the mode of motion of the uniform chain to the mode of locomotion of a millipede which moves by propagating a nearly periodic succession of pulses down its array of legs. A difference is that the millipede almost certainly controls the pulses using a neurological mechanism whereas our chain settles spontaneously into the ordered state and does not use an extrinsic controlling mechanism. The common feature may be that the ordered correlated motion is probably the most efficient means of transporting the system (or the animal) in the sense that it requires the least power input. The animal has obviously evolved in a Darwinian fashion so that it moves in the most energy-efficient manner; the system moves chaotically initially sampling its phase space in a quasiergodic manner and eventually selects a mode of motion in which the power input just matches the power expended dissipatively. Once the system has found this efficient mode it cannot escape into a less efficient mode unless the power input exceeds the power dissipated and some elastic energy builds up in the chain to be expended chaotically later.

[1] R. Burridge and L. Knopoff, Bull. Seismol. Soc. Am. **57**, 341 (1967).

[2] J. M. Carlson and J. S. Langer, Phys. Rev. Lett. **62**, 2632 (1989); Phys. Rev. A **40**, 6470 (1989).

[3] J. Schmittbuhl, J. P. Vilotte, and S. Roux, Europhys. Lett. **21**, 374 (1993).

[4] M. A. Rubio, B. J. Gluckman, A. Dougherty, and J. P. Gollub, Phys. Rev. A **43**, 811 (1991).

[5] D. P. Vallette and J. P. Gollub, Phys. Rev. E **47**, 820 (1993).

[6] J. Huang and D. L. Turcotte, Nature **348**, 234 (1990).